# 1 Limits and Their Properties

# 1.3 Evaluating Limits Analytically

# Objectives

- Evaluate a limit using properties of limits.
- Develop and use a strategy for finding limits.
- Evaluate a limit using the dividing out technique.
- Evaluate a limit using the rationalizing technique.
- Evaluate a limit using the Squeeze Theorem.

# Properties of Limits

# Properties of Limits

The limit of *f*(*x*) as *x* approaches *c* does not depend on the value of *f* at *x* = *c*. It may happen, however, that the limit is precisely *f*(*c*). In such cases, we can evaluate the limit by **direct substitution**. That is,

$$\lim_{x \to c} f(x) = f(c). \qquad \text{Substitute } c \text{ for } x.$$

Such *well-behaved* functions are **continuous at *c***.

**THEOREM 1.1 Some Basic Limits**

Let $b$ and $c$ be real numbers, and let $n$ be a positive integer.

**1.** $\lim_{x \to c} b = b$ **2.** $\lim_{x \to c} x = c$ **3.** $\lim_{x \to c} x^n = c^n$

# Example 1 – *Evaluating Basic Limits*

**a.** $\lim_{x \to 2} 3 = 3$

**b.** $\lim_{x \to -4} x = -4$

**c.** $\lim_{x \to 2} x^2 = 2^2 = 4$

# Properties of Limits

**THEOREM 1.2 Properties of Limits**

Let $b$ and $c$ be real numbers, let $n$ be a positive integer, and let $f$ and $g$ be functions with the limits

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = K.$$

**1.** Scalar multiple: $\lim_{x \to c} [b\,f(x)] = bL$

**2.** Sum or difference: $\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$

**3.** Product: $\lim_{x \to c} [f(x)g(x)] = LK$

**4.** Quotient: $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}, \quad K \neq 0$

**5.** Power: $\lim_{x \to c} [f(x)]^n = L^n$

# Example 2 – *The Limit of a Polynomial*

Find the limit: $\lim_{x \to 2} (4x^2 + 3)$.

Solution:

$$\lim_{x \to 2} (4x^2 + 3) = \lim_{x \to 2} 4x^2 + \lim_{x \to 2} 3 \qquad \text{Property 2, Theorem 1.2}$$

$$= 4\left(\lim_{x \to 2} x^2\right) + \lim_{x \to 2} 3 \qquad \text{Property 1, Theorem 1.2}$$

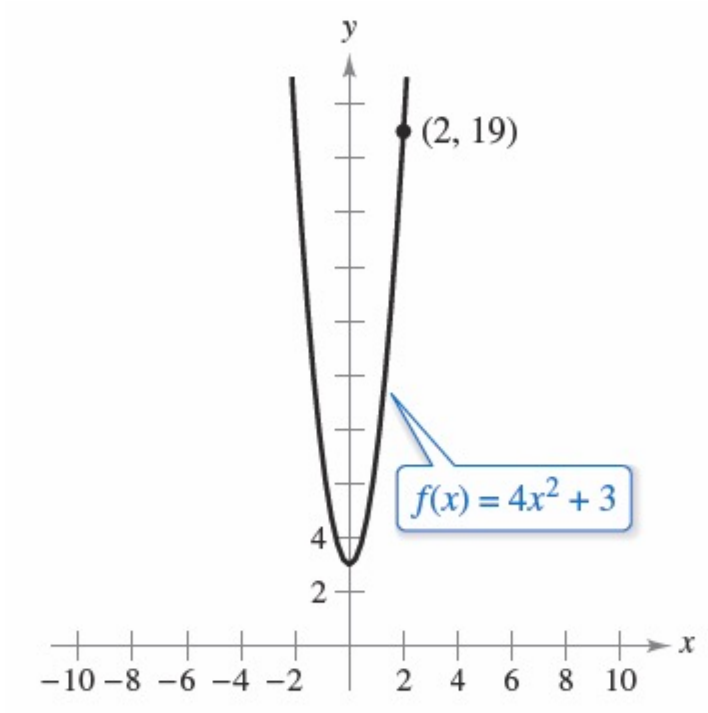
$$= 4(2^2) + 3 \qquad \text{Properties 1 and 3, Theorem 1.1}$$

$$= 19 \qquad \text{Simplify.}$$

# Example 2 – *Solution*

cont'd

This limit is reinforced by the graph of $f(x) = 4x^2 + 3$ shown in Figure 1.17.




The limit of $f(x)$ as $x$ approaches 2 is 19.

**Figure 1.17**

# Properties of Limits

The limit (as $x$ approaches 2) of the *polynomial function* $p(x) = 4x^2 + 3$ is simply the value of $p$ at $x = 2$.

$$\lim_{x \to 2} p(x) = p(2) = 4(2^2) + 3 = 19$$

This *direct substitution* property is valid for all polynomial and rational functions with nonzero denominators.

# Properties of Limits

**THEOREM 1.3 Limits of Polynomial and Rational Functions**

If $p$ is a polynomial function and $c$ is a real number, then

$$\lim_{x \to c} p(x) = p(c).$$

If $r$ is a rational function given by $r(x) = p(x)/q(x)$ and $c$ is a real number such that $q(c) \neq 0$, then

$$\lim_{x \to c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

# Example 3 – *The Limit of a Rational Function*

Find the limit: $\lim_{x \to 1} \frac{x^2 + x + 2}{x + 1}$.

Solution:

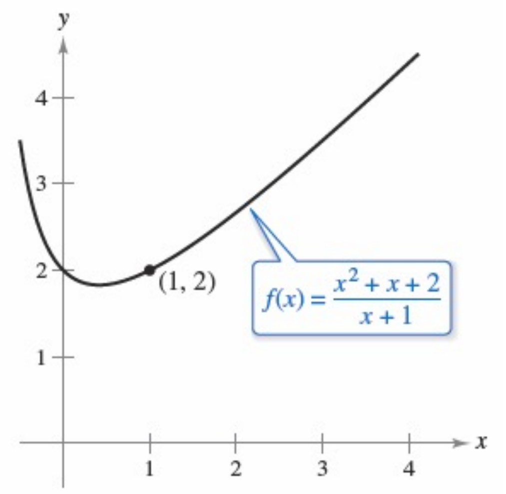
Because the denominator is not 0 when *x* = 1, you can apply Theorem 1.3 to obtain

$$\lim_{x \to 1} \frac{x^2 + x + 2}{x + 1} = \frac{1^2 + 1 + 2}{1 + 1}$$

$$= \frac{4}{2}$$

$$= 2.$$

See Figure 1.18.




The limit of $f(x)$ as $x$ approaches 1 is 2.

**Figure 1.18**

# Properties of Limits

Polynomial functions and rational functions are two of the three basic types of algebraic functions.

The next theorem deals with the limit of the third type of algebraic function—one that involves a radical.

**THEOREM 1.4 The Limit of a Function Involving a Radical**

Let $n$ be a positive integer. The limit below is valid for all $c$ when $n$ is odd, and is valid for $c > 0$ when $n$ is even.

$$\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$$

A proof of this theorem is given in Appendix A.

# Properties of Limits

The next theorem greatly expands your ability to evaluate limits because it shows how to analyze the limit of a composite function.

**THEOREM 1.5 The Limit of a Composite Function**

If $f$ and $g$ are functions such that $\lim_{x \to c} g(x) = L$ and $\lim_{x \to L} f(x) = f(L)$, then

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(L).$$

# Example 4 – *The Limit of a Composite Function*

Find the limit.

**a.** $\lim_{x \to 0} \sqrt{x^2 + 4}$ **b.** $\lim_{x \to 3} \sqrt[3]{2x^2 - 10}$

Solution:

**a.** Because

$$\lim_{x \to 0} (x^2 + 4) = 0^2 + 4 = 4 \quad \text{and} \quad \lim_{x \to 4} \sqrt{x} = \sqrt{4} = 2$$

you can conclude that

$$\lim_{x \to 0} \sqrt{x^2 + 4} = \sqrt{4} = 2.$$

# Example 4 – *Solution*

cont'd

**b**. Because

$$\lim_{x\to 3} (2x^2 - 10) = 2(3^2) - 10 = 8 \quad \text{and} \quad \lim_{x\to 8} \sqrt[3]{x} = \sqrt[3]{8} = 2$$

you can conclude that

$$\lim_{x\to 3} \sqrt[3]{2x^2 - 10} = \sqrt[3]{8} = 2.$$

# Properties of Limits

You have seen that the limits of many algebraic functions can be evaluated by direct substitution. The six basic trigonometric functions also exhibit this desirable quality, as shown in the below theorem.

**THEOREM 1.6 Limits of Trigonometric Functions**

Let $c$ be a real number in the domain of the given trigonometric function.

1. $\lim_{x \to c} \sin x = \sin c$
2. $\lim_{x \to c} \cos x = \cos c$
3. $\lim_{x \to c} \tan x = \tan c$
4. $\lim_{x \to c} \cot x = \cot c$
5. $\lim_{x \to c} \sec x = \sec c$
6. $\lim_{x \to c} \csc x = \csc c$

# Example 5 – *Limits of Trigonometric Functions*

**a.** $\lim_{x \to 0} \tan x = \tan(0) = 0$

**b.** $\lim_{x \to \pi} (x \cos x) = \left(\lim_{x \to \pi} x\right)\left(\lim_{x \to \pi} \cos x\right) = \pi \cos(\pi) = -\pi$

**c.** $\lim_{x \to 0} \sin^2 x = \lim_{x \to 0} (\sin x)^2 = 0^2 = 0$

# A Strategy for Finding Limits

# A Strategy for Finding Limits

You studied several types of functions whose limits can be evaluated by direct substitution. This knowledge, together with the next theorem, can be used to develop a strategy for finding limits.

**THEOREM 1.7 Functions That Agree at All but One Point**

Let $c$ be a real number, and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing $c$. If the limit of $g(x)$ as $x$ approaches $c$ exists, then the limit of $f(x)$ also exists and

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x).$$

# Example 6 – *Finding the Limit of a Function*

Find the limit.

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$$

Solution:

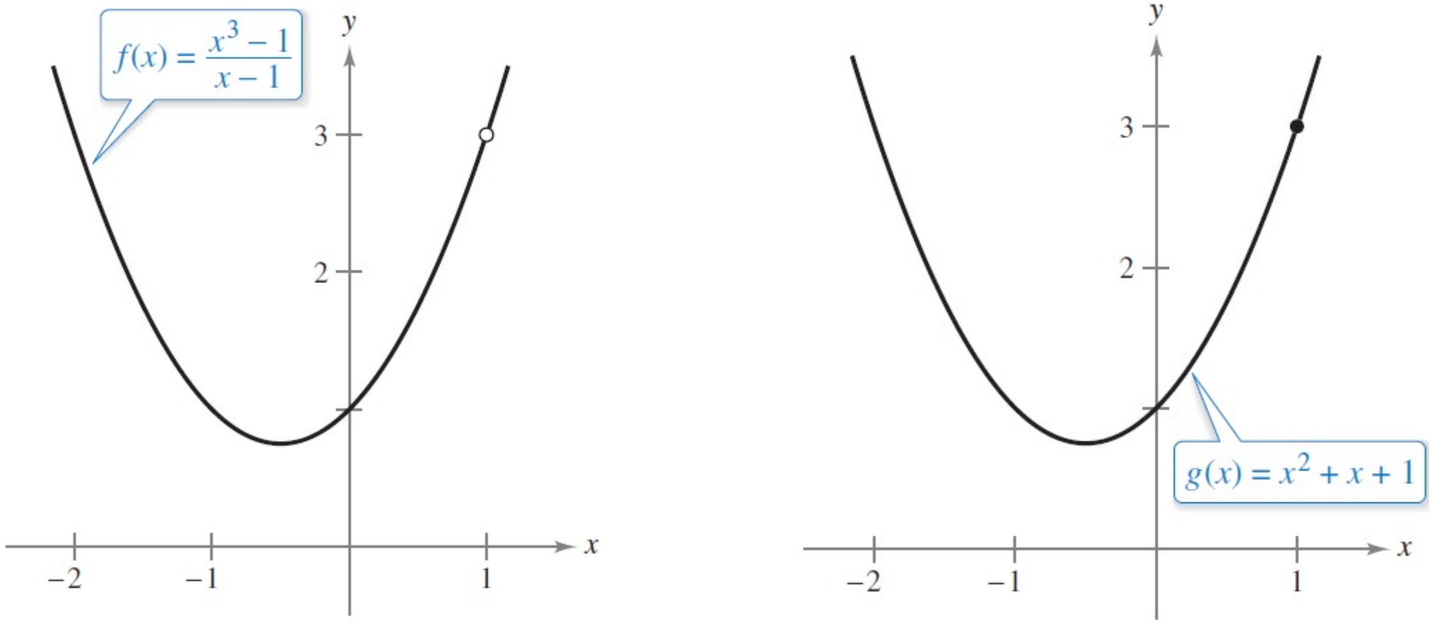
Let $f(x) = (x^3 - 1)/(x - 1)$

By factoring and dividing out common factors, you can rewrite $f$ as

$$f(x) = \frac{\cancel{(x-1)}(x^2 + x + 1)}{\cancel{(x-1)}} = x^2 + x + 1 = g(x), \quad x \neq 1.$$

# Example 6 – *Solution* cont'd

So, for all $x$-values other than $x = 1$, the functions $f$ and $g$ agree, as shown in Figure 1.19.




*f* and *g* agree at all but one point.

**Figure 1.19**

# Example 6 – *Solution*

cont'd

Because $\lim_{x \to 1} g(x)$ exists, you can apply Theorem 1.7 to conclude that *f* and *g* have the same limit at *x* = 1.

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x^2 + x + 1)}{x - 1} \qquad \text{Factor.}$$

$$= \lim_{x \to 1} \frac{\cancel{(x - 1)}(x^2 + x + 1)}{\cancel{x - 1}} \qquad \text{Divide out like factors.}$$

$$= \lim_{x \to 1}(x^2 + x + 1) \qquad \text{Apply Theorem 1.7.}$$

$$= 1^2 + 1 + 1 \qquad \text{Use direct substitution.}$$

$$= 3 \qquad \text{Simplify.}$$

# A Strategy for Finding Limits

**A Strategy for Finding Limits**

1. Learn to recognize which limits can be evaluated by direct substitution. (These limits are listed in Theorems 1.1 through 1.6.)
2. When the limit of $f(x)$ as $x$ approaches $c$ *cannot* be evaluated by direct substitution, try to find a function $g$ that agrees with $f$ for all $x$ other than $x = c$. [Choose $g$ such that the limit of $g(x)$ *can* be evaluated by direct substitution.] Then apply Theorem 1.7 to conclude *analytically* that

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = g(c).$$

3. Use a *graph* or *table* to reinforce your conclusion.

Dividing Out Technique

# Dividing Out Technique

One procedure for finding a limit analytically is the **dividing out technique**. This technique involves diving out common factors.

# Example 7 – *Dividing Out Technique*

Find the limit: $\lim_{x \to -3} \dfrac{x^2 + x - 6}{x + 3}$.

Solution:

Although you are taking the limit of a rational function, you *cannot* apply Theorem 1.3 because the limit of the denominator is 0.

$$\lim_{x \to -3} (x^2 + x - 6) = 0$$

$$\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3}$$

Direct substitution fails.

$$\lim_{x \to -3} (x + 3) = 0$$

# Example 7 – *Solution*

cont'd

Because the limit of the numerator is also 0, the numerator and denominator have a *common factor* of ($x$ + 3).

So, for all $x \neq -3$, you can divide out this factor to obtain

$$f(x) = \frac{x^2 + x - 6}{x + 3} = \frac{\cancel{(x + 3)}(x - 2)}{\cancel{x + 3}} = x - 2 = g(x), \quad x \neq -3.$$

Using Theorem 1.7, it follows that

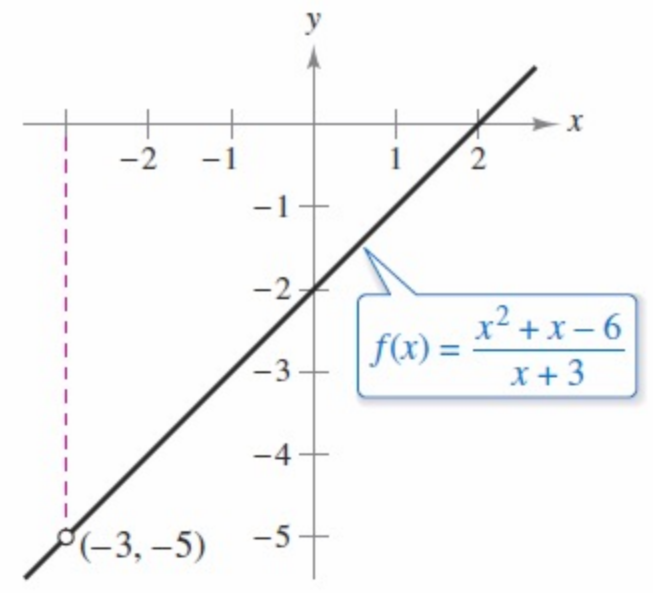
$$\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3} = \lim_{x \to -3} (x - 2) \qquad \text{Apply Theorem 1.7.}$$

$$= -5. \qquad \text{Use direct substitution.}$$

# Example 7 – *Solution*

cont'd

This result is shown graphically in Figure 1.20. Note that the graph of the function *f* coincides with the graph of the function $g(x) = x - 2$, except that the graph of *f* has a hole at the point (–3, –5).




*f* is undefined when $x = -3$. The limit of $f(x)$ as $x$ approaches $-3$ is $-5$.

**Figure 1.20**

# Dividing Out Technique

In Example 7, direct substitution produced the meaningless fractional form 0/0. An expression such as 0/0 is called an **indeterminate form** because you cannot (from the form alone) determine the limit.

When you try to evaluate a limit and encounter this form, remember that you must rewrite the fraction so that the new denominator does not have 0 as its limit. One way to do this is to *divide out common factors*.

# Rationalizing Technique

# Rationalizing Technique

Another way to find a limit analytically is the **rationalizing technique**, which involves rationalizing either the numerator or denominator of a fractional expression. We know that rationalizing the numerator (denominator) means multiplying the numerator and denominator by the conjugate of the numerator (denominator).

For instance, to rationalize the numerator of

$$\frac{\sqrt{x} + 4}{x}$$

multiply the numerator and denominator by the conjugate of $\sqrt{x} + 4$, which is $\sqrt{x} - 4$.

# Example 8 – *Rationalizing Technique*

Find the limit: $$\lim_{x \to 0} \frac{\sqrt{x+1}-1}{x}.$$

Solution:

By direct substitution, you obtain the indeterminate form 0/0.

$$\begin{tikzcd}[column sep=large]
 & \lim\limits_{x \to 0} \left(\sqrt{x+1}-1\right) = 0 \\
\lim\limits_{x \to 0} \dfrac{\sqrt{x+1}-1}{x} \arrow[ur] \arrow[dr] & \\
 & \lim\limits_{x \to 0} x = 0
\end{tikzcd}$$

Direct substitution fails.

# Example 8 – *Solution*

cont'd

In this case, you can rewrite the fraction by rationalizing the numerator.

$$\frac{\sqrt{x+1}-1}{x} = \left(\frac{\sqrt{x+1}-1}{x}\right)\left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right)$$

$$= \frac{(x+1)-1}{x\left(\sqrt{x+1}+1\right)}$$

$$= \frac{\cancel{x}}{\cancel{x}\left(\sqrt{x+1}+1\right)}$$

$$= \frac{1}{\sqrt{x+1}+1}, \qquad x \neq 0$$

# Example 8 – *Solution*

cont'd

Now, using Theorem 1.7, you can evaluate the limit as shown.

$$\lim_{x \to 0} \frac{\sqrt{x+1}-1}{x} = \lim_{x \to 0} \frac{1}{\sqrt{x+1}+1}$$

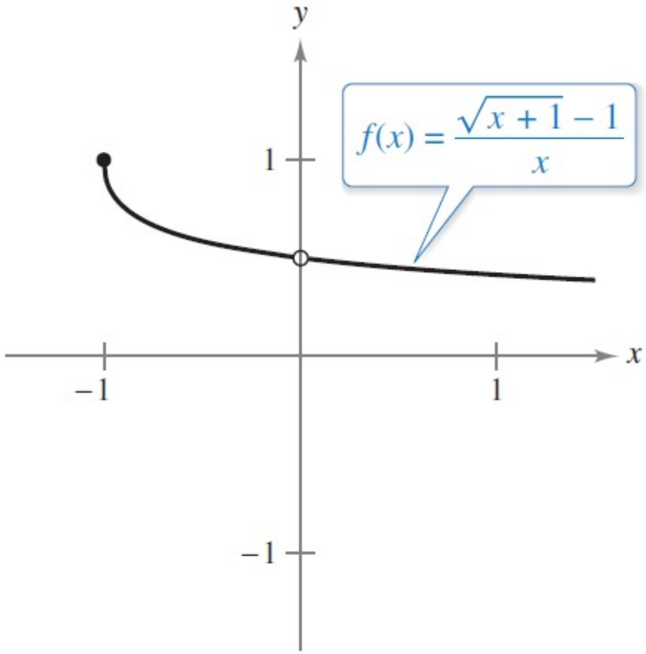
$$= \frac{1}{1+1}$$

$$= \frac{1}{2}$$

# Example 8 – *Solution*

cont'd

A table or a graph can reinforce your conclusion that the limit is 1/2 . (See Figure 1.21.)




The limit of $f(x)$ as $x$ approaches 0 is $\frac{1}{2}$.

**Figure 1.21**

# Example 8 – *Solution*

cont'd

$x$ approaches 0 from the left.

$x$ approaches 0 from the right.

| $x$ | $-0.25$ | $-0.1$ | $-0.01$ | $-0.001$ | 0 | 0.001 | 0.01 | 0.1 | 0.25 |
|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 0.5359 | 0.5132 | 0.5013 | 0.5001 | ? | 0.4999 | 0.4988 | 0.4881 | 0.4721 |

$f(x)$ approaches 0.5.

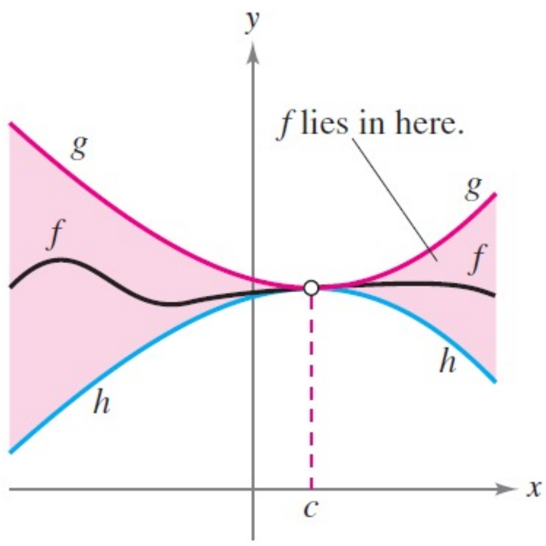
$f(x)$ approaches 0.5.

# The Squeeze Theorem

# The Squeeze Theorem

The next theorem concerns the limit of a function that is squeezed between two other functions, each of which has the same limit at a given *x*-value, as shown in Figure 1.22

$h(x) \le f(x) \le g(x)$




The Squeeze Theorem

**Figure 1.22**

# The Squeeze Theorem

**THEOREM 1.8 The Squeeze Theorem**

If $h(x) \le f(x) \le g(x)$ for all $x$ in an open interval containing $c$, except possibly at $c$ itself, and if

$$\lim_{x \to c} h(x) = L = \lim_{x \to c} g(x)$$

then $\lim_{x \to c} f(x)$ exists and is equal to $L$.

The Squeeze Theorem is also called the Sandwich Theorem or the Pinching Theorem.

**THEOREM 1.9 Two Special Trigonometric Limits**

**1.** $\displaystyle \lim_{x \to 0} \frac{\sin x}{x} = 1$ **2.** $\displaystyle \lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

## Example 9 – *A Limit Involving a Trigonometric Function*

Find the limit: $\lim_{x \to 0} \frac{\tan x}{x}$.

Solution:

Direct substitution yields the indeterminate form 0/0.

To solve this problem, you can write tan *x* as (sin *x*)/(cos *x*) and obtain

$$\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \left(\frac{\sin x}{x}\right)\left(\frac{1}{\cos x}\right).$$

# Example 9 – *Solution*

cont'd

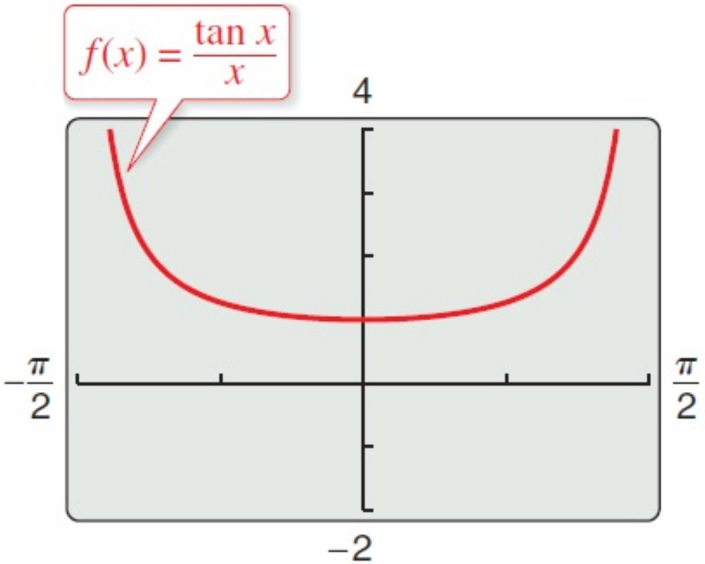
$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \text{and} \qquad \lim_{x \to 0} \frac{1}{\cos x} = 1$$

$$\begin{aligned}
\lim_{x \to 0} \frac{\tan x}{x} &= \left(\lim_{x \to 0} \frac{\sin x}{x}\right)\left(\lim_{x \to 0} \frac{1}{\cos x}\right) \\
&= (1)(1) \\
&= 1.
\end{aligned}$$

# Example 9 – *Solution*

cont'd

(See Figure 1.24.)




The limit of $f(x)$ as $x$ approaches 0 is 1.

**Figure 1.24**